BAYESIAN ESTIMATION OF MULTI-OBJECT SYSTEMS WITH INDEPENDENTLY IDENTICALLY DISTRIBUTED CORRELATIONS

Jeremie Houssineau and Daniel E. Clark

School of Engineering and Physical Sciences, Heriot-Watt University, Edinburgh EH14 4AS

ABSTRACT

Recent generalisations of stochastic filtering methods to multi-object systems have become very popular for solving multi-target tracking problems over the last decade. However, there was previously no general means of introducing correlations between objects. In this article, we investigate generalisations of such multi-object filters for systems where there may be dependencies between objects. Determining probability and factorial moment densities is facilitated by the use of a recent result in variational calculus, a general form of Faà di Bruno's formula. The result is illustrated through the Probability Hypothesis Density (PHD) filter, as a first-order moment example of the general form.

1. INTRODUCTION

Stochastic filtering methods have played a fundamental role in estimation and control theory and its applications. The extension of stochastic filtering methods to systems of multiple objects is a relatively recent development, developed by Mahler [1] as a means of identifying and tracking an unknown number of objects in aerospace applications. Multi-object filters have been based on the stochastic population process framework proposed by Moyal [2], and Mahler's derivation of Bayes' theorem for point processes [3].This methodology is becoming established as the dominant framework for solving multi-target tracking problems within the sensor fusion literature, e.g. [4]. In this framework, however, the dynamics and observations of objects are assumed to be independent, and there is currently no general framework for modelling systems with correlations between objects.

A step towards introducing correlations into multi-object filtering was proposed in [5] based on the Gauss-Poisson process [6]. However, the result is quite involved and highlights the difficulty of determining the general structure of multi-object systems from functional representations without a general mechanism for doing so. In this article, we introduce such a mechanism by using the general higher-order chain rule for variational calculus that was recently derived [7]. This result enables us to determine the Janossy and factorial moment densities of Bayesian estimation equations for different models [8].

2. DIFFERENTIALS

To make use of the generating functional in the following sections, we require the notion of differentials. Since the generating functional will be defined on a Banach space, the generality of the Gâteaux differential is needed. However, we use a slightly restricted form of Gâteaux differential, known as the chain differential [9], in order that we can determine a chain rule [7]. Following this, we describe the general higher-order chain rule. In this section, X, Y and Z denote topological vector spaces.

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DEFINITION 1 (Chain differential, from [9]). *The function $f : X \to Y$ has a chain differential $\delta f(x;\eta)$ at x in the direction η if, for any sequence $\eta_n \to \eta \in X$, and any sequence of real numbers $\theta_n \to 0$, it holds that*

$$\delta f(x;\eta) = \lim_{n\to\infty} \frac{1}{\theta_n}\left(f(x+\theta_n\eta_n) - f(x)\right). \tag{1}$$

The n^{th}-order chain differential can be defined recursively as

$$\delta^n f(x;\eta_1,\dots,\eta_n) = \delta\left(\delta^{n-1} f(x;\eta_1,\dots,\eta_{n-1});\eta_n\right).$$

THEOREM 1 (General higher-order chain rule, from [7]). *Assuming that $g : X \to Y$ has higher order chain differentials in any number of directions in the set $\{\eta_1,\dots,\eta_n\} \in X^n$ and that $f : Y \to Z$ has higher order chain differentials in any number of directions in the set $\{\delta^m g(x;S_m)\}_{m=1:n}$, $S_m \subseteq \{\eta_1,\dots,\eta_n\}$. Assuming additionally that for all $1 \le m \le n$, $\delta^m f(y;\xi_1,\dots,\xi_m)$ is continuous on an open set $\Omega \subseteq Y^{m+1}$ and linear w.r.t. the directions $\xi_1,\dots,\xi_m$, the n^{th}-order variation of composition $f \circ g$ in directions $\eta_1,\dots,\eta_n$ at point $x \in X$ is given by*

$$\delta^n(f\circ g)(x;\eta_1,\dots,\eta_n) = \sum_{\pi\in\Pi(\eta_1,\dots,\eta_n)} \delta^{|\pi|} f\left(g(x);\delta^{|\omega|} g(x;\xi:\xi\in\omega):\omega\in\pi\right),$$

where $\Pi(A)$ represents the set of partitions of the discrete set A, $|\pi|$ denotes the cardinality of the set π and where $h(y:y\in\pi) = h(y_1,\dots,y_m)$ when $\pi = \{y_1,\dots,y_m\}$.

3. POINT PROCESSES

In this section we introduce the representation used for providing the stochastic description of multiple object, i.e. point processes [10].

Our knowledge about systems of multiple objects is intrinsically stochastic and can be represented by a probability space $(\Omega,\mathcal{F},\mathbb{P})$. The knowledge about the probability of any possible event about any part of the stochastic population is encoded into the probability space $(\Omega,\mathcal{F},\mathbb{P})$. However nothing is known about this space and the population is represented into another measurable space $(E,\mathcal{E})$ where the important characteristics of the population are only represented with points in a Polish space $\mathbf{X}$, e.g. $\mathbb{R}^d$ for some $d\in\mathbb{N}$ including positions, velocities or sizes of objects. The space E can then be considered as the space of sequences $\boldsymbol{x} = (\boldsymbol{x}_i)_{i=1}^n$ of points in $\mathbf{X}$ [2, 11]. Details about the construction of the σ-algebra $\mathcal{E}$ on E can be found in [2] or in [12]. A point process Φ is the measurable mapping between the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the measurable space $(E, \mathcal{E})$. The law of the point process Φ on E is denoted P_Φ. The probability generating functional (p.g.fl.) is an important tool for point process theory [2] and is defined as follows.

DEFINITION 2 (Probability generating functional). *Let $\mathcal{U}(\mathbf{X})$ be the set of bounded measurable functions on $\mathbf{X}$ such that $u \in \mathcal{U}(\mathbf{X})$ satisfies $\sup_{x\in\mathbf{X}} |u(x)| \leq 1$. Let $\mathcal{V}(\mathbf{X})$ be the space of functions $v \in \mathcal{U}(\mathbf{X})$ such that $1 - v$ is vanishing outside some bounded set and is satisfying $0 \leq v(x) \leq 1$, for any $x \in \mathbf{X}$. The probability generating functional G_Φ of a process Φ can be written for $v \in \mathcal{V}(\mathbf{X})$ as*

$$\begin{aligned} G_\Phi(v) &= \sum_{n\geq 0} \int_{\mathbf{X}^n} v(\boldsymbol{x}_1)\dots v(\boldsymbol{x}_n) P_\Phi^{(n)}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ &= \int_E \Big[\prod_{x\in\boldsymbol{x}} v(x)\Big] P_\Phi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \end{aligned}$$

where $P_\Phi^{(n)}$ is the projection of probability distribution P_Φ on $\mathbf{X}^n$ and gives the probability of having n points as well as their distribution.

The superscript of the projection $P^{(n)}$ will be dropped when it is not informative such as in $P^{(|\boldsymbol{x}|)}(\boldsymbol{x})$, as opposed to $P^{(n)}(\boldsymbol{x})$.

DEFINITION 3. *The n^{th}-order Janossy density $J_\Phi^{(n)}$ and the n^{th}-order factorial moment density $M_\Phi^{(n)}$ are defined on $\mathbf{X}^n$ as follows*

$$\begin{aligned} J_\Phi^{(n)}(\boldsymbol{x}) &= \sum_{\sigma\in S_n} P_\Phi^{(n)}(\boldsymbol{x}_{\sigma(1)}, \dots, \boldsymbol{x}_{\sigma(k)}), \\ \int_{\mathbf{X}^n} f(\boldsymbol{x}) M_\Phi^{(n)}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= \sum_{k\geq n} \int_{\mathbf{X}^k} \sum_{\substack{\boldsymbol{y}\subseteq\boldsymbol{x} \\ |\boldsymbol{y}|=n}} f(\boldsymbol{y}) P^{(k)}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \end{aligned}$$

where S_n is the set of all permutations of the set $\{1, \dots, n\}$.

The Janossy density has a simple meaning, as $J_\Phi(\boldsymbol{x})$ can be interpreted as the probability of having one object at each of the points in $\boldsymbol{x}$. Even though the definition of the n^{th}-order factorial moment density is more involved, we can see that this quantity is related to the average behaviour of any n points in the realisations of the process Φ. The first factorial moment is also the mean of the process, while the second factorial moment represent the joint statistics of any two points in the point process Φ. The latter must not be confused with the standard second moment from which the variance of the process can be deduced [13] and [14].

Moyal [2] also shown that the Janossy density and factorial moment density are recovered from the p.g.fl. with

$$J_\Phi^{(n)}(\boldsymbol{x}) = \delta^n G_\Phi(v; \delta_{\boldsymbol{x}_1}, \dots, \delta_{\boldsymbol{x}_n})|_{v=0}, \tag{2}$$

$$M_\Phi^{(n)}(\boldsymbol{x}) = \delta^n G_\Phi(v; \delta_{\boldsymbol{x}_1}, \dots, \delta_{\boldsymbol{x}_n})|_{v=1}. \tag{3}$$

Most often, the chain differentials will be in the directions of δ-functions, so that, for notational convenience, we will denote $\Delta_{\boldsymbol{x}}$ the list $\delta_{\boldsymbol{x}_1}, \dots, \delta_{\boldsymbol{x}_n}$, for any $\boldsymbol{x} \in \mathbf{X}^n$ and any $n \in \mathbb{N}$. Also, in order to conveniently describe factorial moments, we introduce the function $C_{n|k}$ on $\mathbf{X}^n \times \mathbf{X}^k$, for any $n, k \in \mathbb{N}$ such that

$$C_{n|k}(\boldsymbol{x}|\boldsymbol{y}) = \delta^n \bigg(\prod_{y\in\boldsymbol{y}} v(y); \Delta_{\boldsymbol{x}}\bigg)\bigg|_{v=1}, \tag{4}$$

Note that $C_{n|k}$ vanishes when $n > k$. In the next section, the tools developed here for describing general point processes are used to characterise interacting point processes.

4. POINT PROCESSES WITH CORRELATIONS

In this section we describe a class of point processes for modelling interactions, first within a point process and then with interactions between two point processes, by introducing specific joint point processes. We denote $E^* = E \setminus \mathbf{X}^0$ the space of non-empty sequences of points in $\mathbf{X}$.

DEFINITION 4. *Let $\{K_\Psi^{(n)}\}_{n\geq 1}$ be a family of densities, which might not be probability densities, then the Khinchin generating functional $G_\Psi : \mathcal{V}(\mathbf{X}) \to \mathbb{R}$ is defined as*

$$G_\Psi(v) = \exp\left(L_\Psi(v)\right)$$

where, denoting $K_\Psi^{(0)} = \int_{E^*} K_\Psi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$, L_Ψ is expressed as

$$L_\Psi(v) = -K_\Psi^{(0)} + \int_{E^*} \Big[\prod_{x\in\boldsymbol{x}} v(x)\Big] K_\Psi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

Using Theorem 9.4.V from [10, Vol.II], we find that the functional G_Ψ defines a unique point process.

DEFINITION 5. *The point process with p.g.fl. G_Ψ is called a Khinchin point process.*

This class of processes have been introduced in [15]. To better understand the characteristics of a Khinchin point process, it is useful to determine the Janossy and factorial moment densities of such a process.

PROPOSITION 1. *The n^{th}-order Janossy density $J_\Psi^{(n)}$ and the n^{th}-order factorial moment density $M_\Psi^{(n)}$ of a Khinchin process Ψ are determined with*

$$\begin{aligned} J_\Psi^{(n)}(\boldsymbol{x}) &= \exp\Big(-K_\Psi^{(0)}\Big) \sum_{\pi\in\Pi(\boldsymbol{x})} \prod_{\omega\in\pi} |\omega|! K_\Psi(\omega) \\ M_\Psi^{(n)}(\boldsymbol{x}) &= \sum_{\pi\in\Pi(\boldsymbol{x})} \prod_{\omega\in\pi} Q_\Psi(\omega), \end{aligned}$$

where $Q_\Psi^{(n)}(\boldsymbol{x}) = \delta^n L_\Psi(v; \Delta_{\boldsymbol{x}})|_{v=1}$.

Proof. It is easily shown that the n^{th} chain differential of the exponential function takes the form

$$\delta^n \exp(x; \eta_1, \dots, \eta_n) = \exp(x)\eta_1 \dots \eta_n.$$

The result is found by using (2), (3) and Theorem 1, as well as by noting that $L_\Psi(0) = -K_\Psi^{(0)}$, $L_\Psi(1) = 0$, and that the n^{th} chain differential of L_Ψ at point $v = 0$ is found to be $\delta^n L_\Psi(v; \Delta_{\boldsymbol{x}})|_{v=0} = K_\Psi^{(n)}(\boldsymbol{x})$. ☐

When analysing the form of the Janossy densities J_Ψ of the Khinchin process Ψ, we see that this process has an i.i.d. distribution for groups of points of the same size, as depicted in Figure 1. The n^{th}-order factorial moment is more difficult to analyse, however, assuming $K_\Psi^{(n)} = 0$ for any $n > 2$, we obtain a Gauss-Poisson point process, and the second factorial moment is found to be

$$M_\Psi^{(2)}(x_1, x_2) = 2K_\Psi^{(2)}(x_1, x_2) + M_\Psi^{(1)}(x_1) M_\Psi^{(1)}(x_2).$$

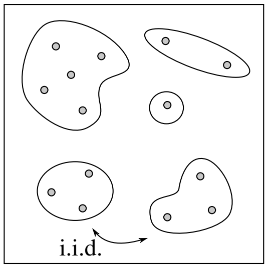

Fig. 1. A realisation of a Khinchin process, with correlated points

We can recognise the form of a covariance and find that $\mathrm{cov}_\Psi(x_1, x_2) = 2K_\Psi^{(2)}(x_1, x_2)$. In other words, the density $K_\Psi^{(2)}$ of a Gauss-Poisson point process directly represent the correlations in the process.

Equipped with Khinchin point processes, one can model interactions and independence in a point process. We now consider another point processes Ψ' from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(E', \mathcal{E}')$ with points in the Polish space $\mathbf{X}'$. The subscript $\diamond$ will be used for quantities representing both Ψ and Ψ' while $\triangleright$ and $\triangleleft$ will stand for the conditionals $\Psi'|\Psi$ and $\Psi|\Psi'$. Also, the joint space $E^*_\diamond$ is defined as $E^*_\diamond = (E \times E') \setminus (\mathbf{X}^0 \times \mathbf{X}'^0)$.

DEFINITION 6. *A joint Khinchin process can be defined through its p.g.fl. $G_\diamond$ on $\mathcal{V}(\mathbf{X}) \times \mathcal{V}(\mathbf{X}')$ expressed as*

$$G_\diamond(v, v') = \exp\Big(-K_\diamond^{(0,0)}\Big)$$
$$\times \exp\bigg(\int_{E^*_\diamond} \Big[\prod_{x\in\boldsymbol{x}} v(x)\Big]\Big[\prod_{x'\in\boldsymbol{x}'} v'(x')\Big] K_\diamond(\boldsymbol{x}, \boldsymbol{x}')\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}'\bigg),$$

where $K_\diamond^{(0,0)}$ is a normalising constant that ensures that $G_\diamond(1,1) = 1$, i.e.

$$K_\diamond^{(0,0)} = \int_{E^*_\diamond} K_\diamond(\boldsymbol{x}, \boldsymbol{x}')\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x}'.$$

Using the generality of the chain rule described in Theorem 1, it is easy to find the Janossy densities of a joint Khinchin point process.

PROPOSITION 2. *The Janossy densities of a bivariate Khinchin point process is*

$$J_\diamond(\boldsymbol{x}, \boldsymbol{x}') =$$
$$\exp\big(-K_\diamond^{(0,0)}\big) \sum_{\pi\in\Pi(\boldsymbol{x}\cup\boldsymbol{x}')} \prod_{\omega\in\pi} |\omega_{\boldsymbol{x}}|!|\omega_{\boldsymbol{x}'}|!K_\diamond(\omega_{\boldsymbol{x}}, \omega_{\boldsymbol{x}'}),$$

where $\omega_{\boldsymbol{x}}$ stands for $\omega \cap \boldsymbol{x}$.

Proof. We can recover the Janossy density from $G_\diamond$ via differentiation in both variables, i.e.

$$J_\diamond^{(n,m)}(\boldsymbol{x}, \boldsymbol{x}') = \delta^{n+m} G_\diamond\big(v, v'; \Delta_{\boldsymbol{x}}, \Delta_{\boldsymbol{x}'}\big)\big|_{v=v'=0}.$$

□

This approach to joint point process will be useful in the next section for handling Bayesian estimation for multiple objects.

5. BAYESIAN ESTIMATION

Considering a joint Khinchin process as in Section 4, it is of interest to describe how the Janossy and factorial moment densities of Ψ are updated when obtaining a realisation of the point process Ψ'. Note that the joint densities $K_\diamond$ can be expressed as

$$K_\diamond(\boldsymbol{x}, \boldsymbol{x}') = L_\triangleright(\boldsymbol{x}'|\boldsymbol{x})K_\Psi(\boldsymbol{x}),$$

where $L_\triangleright$ is a likelihood. We additionally define $\ell_\triangleright^{(n)} = n!L_\triangleright^{(n)}$.

THEOREM 2. *The Janossy and factorial moment densities of an updated Khinchin process via Bayes' theorem is computed with*

$$J_\triangleleft(\boldsymbol{x}|\boldsymbol{x}') \propto \sum_{\pi\in\Pi(\boldsymbol{x}\cup\boldsymbol{x}')} \prod_{\omega\in\pi} |\omega_{\boldsymbol{x}}|!\ell_\triangleright(\omega_{\boldsymbol{x}'}|\omega_{\boldsymbol{x}})K_\Psi(\omega_{\boldsymbol{x}})$$
$$M_\triangleleft(\boldsymbol{x}|\boldsymbol{x}') \propto \sum_{\pi\in\Pi(\boldsymbol{x}\cup\boldsymbol{x}')} \prod_{\omega\in\pi} \int_E C(\omega_{\boldsymbol{x}}|\boldsymbol{y})\ell_\triangleright(\omega_{\boldsymbol{x}'}|\boldsymbol{y})K_\Psi(\boldsymbol{y})\mathrm{d}\boldsymbol{y},$$

where C is as defined in (4) *and the common normalisation term is*

$$\sum_{\pi\in\Pi(\boldsymbol{x}')} \prod_{\omega\in\pi} \int_E \ell_\triangleright(\omega_{\boldsymbol{x}'}|\boldsymbol{y})K_\Psi(\boldsymbol{y})\mathrm{d}\boldsymbol{y}.$$

Proof. The result can be determined directly, for any $k, n \in \mathbb{N}$, from the functional $G_\diamond(v, v')$ with

$$J_\triangleleft^{(k|n)}(\boldsymbol{x}|\boldsymbol{x}') = \frac{\delta^{n+k} G_\diamond(v, v'; \Delta_{\boldsymbol{x}}, \Delta_{\boldsymbol{x}'})|_{v=v'=0}}{\delta^n G_\diamond(1, v'; \Delta_{\boldsymbol{x}'})|_{v'=0}},$$
$$M_\triangleleft^{(k|n)}(\boldsymbol{x}|\boldsymbol{x}') = \frac{\delta^{n+k} G_\diamond(v, v'; \Delta_{\boldsymbol{x}}, \Delta_{\boldsymbol{x}'})|_{v=1,v'=0}}{\delta^n G_\diamond(1, v'; \Delta_{\boldsymbol{x}'})|_{v'=0}},$$

using Theorem 1 on the joint Khinchin process generating functional. □

The Probability Hypothesis Density (PHD) filter [3] was developed as a tractable means of propagating a point process for multiple target tracking applications. A Poisson approximation of the prior population process was made before applying Bayes' theorem in order to derive a closed-form recursion in the first-order factorial moment density.

Example 1 (Multiple-target tracking)**.** *Starting from the Bayes' update for joint Khinchin processes in Theorem 2, written at the first order, the assumptions needed to arrive to the usual PHD filter are*

1. *the densities $K_\Psi^{(n)}$ vanishes for $n > 1$ and $K_\Psi^{(1)} = \lambda P_\Psi^{(1)}$*
2. *no more than one measurement is originated from each object*
3. *measurements are independent of each other.*

The only partitions left in the set $\Pi(x \cup \boldsymbol{x}')$, when $|\boldsymbol{x}'| = n$, are of the form

- *$\{x, \boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_n\}$, i.e. no measurement from a target,*
- *$\{\boldsymbol{x}'_1, \ldots, \{x, \boldsymbol{x}'_i\}, \ldots, \boldsymbol{x}'_n\}$ for $1 \le i \le n$, i.e. one measurement per target.*

Therefore, noting that $J_\triangleright^{(m)}(\cdot|x) = P_\triangleright^{(m)}(\cdot|x)$ when $m \le 1$, the first moment in Theorem 2 becomes the PHD filter update:

$$M_\triangleleft(x|\boldsymbol{x}') =$$
$$P_\triangleright^{(0|1)}(x)M_\Psi(x) + \sum_{x'\in\boldsymbol{x}'} \frac{P_\triangleright(x'|x)M_\Psi(x)}{\nu_{\Psi'}(x') + \int_{\mathbf{X}} P_\triangleright(x'|x)M_\Psi(x)\mathrm{d}x},$$

where the density $\nu_{\Psi'}(x')$ refers to $J_\triangleright^{(1|0)}(x')K_\Psi^{(0)}$ and is the density of the false alarm process, and where $P_\triangleright^{(0,1)}$ is the probability for a object at x to generate no measurement.

The underlying assumption of Bayesian estimation for Khinchin process is that the interactions must be i.i.d., in other words, these interactions have to be the same over the whole state space. Also, similarly to the PHD filter, Khinchin processes only allow for the estimation of global statistics.

6. CONCLUSION

This paper develops a framework for the estimation of a given class of multi-object dynamical systems with correlations, namely, those with identically distributed interactions. The resulting Bayes' theorem has been proved to be a generalisation of the multi-target tracking Probability Hypothesis Density (PHD) filter. The general results involve summations over partitions of measurements and states, and hence approximations will be required when designing practical algorithms to permit tractable implementations. Yet, the approach demonstrated in this article helps understanding the structure of Bayesian estimation for point processes with interactions and establish a useful basis for the derivation of future filters for multiple interacting objects.

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